Logical Theory of Evaluative Linguistic Expressions I

Vilém Novák

University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
30. dubna 22, 701 03 Ostrava 1, Czech Republic
Vilem.Novak@osu.cz

Praha, 22. 10. 2007

IRAFM

Outline

What we are speaking about?

Small, medium, big, twenty five, roughly one hundred, very short, more or less strong, not tall, about twenty five, the sea is deep but not very, roughly small or medium, very roughly strong, weight is small, pressure is very high, extremely rich person

Importance

Mathematical model of the meaning of evaluative expressions can be ranked among the most important contributions of fuzzy logic

L. A. Zadeh

Why?

- Omnipresent in natural language (people need to evaluate); occur in description of any process, decision situation, procedure, characterization of objects, etc.
- Belong to the agenda of fuzzy logic
  Fuzzy IF-THEN rules, Linguistic variable
- They are present in applications of fuzzy logic
  Fuzzy control, decision making, classification, various industrial applications

Importance

Mathematical model of the meaning of evaluative expressions can be ranked among the most important contributions of fuzzy logic

L. A. Zadeh

Why?

- Omnipresent in natural language (people need to evaluate); occur in description of any process, decision situation, procedure, characterization of objects, etc.
- Belong to the agenda of fuzzy logic
  Fuzzy IF-THEN rules, Linguistic variable
- They are present in applications of fuzzy logic
  Fuzzy control, decision making, classification, various industrial applications

Importance

Mathematical model of the meaning of evaluative expressions can be ranked among the most important contributions of fuzzy logic

L. A. Zadeh

Why?

- Omnipresent in natural language (people need to evaluate); occur in description of any process, decision situation, procedure, characterization of objects, etc.
- Belong to the agenda of fuzzy logic
  Fuzzy IF-THEN rules, Linguistic variable
- They are present in applications of fuzzy logic
  Fuzzy control, decision making, classification, various industrial applications

Importance

Mathematical model of the meaning of evaluative expressions can be ranked among the most important contributions of fuzzy logic

L. A. Zadeh

Why?

- Omnipresent in natural language (people need to evaluate); occur in description of any process, decision situation, procedure, characterization of objects, etc.
- Belong to the agenda of fuzzy logic
  Fuzzy IF-THEN rules, Linguistic variable
- They are present in applications of fuzzy logic
  Fuzzy control, decision making, classification, various industrial applications

Importance

Mathematical model of the meaning of evaluative expressions can be ranked among the most important contributions of fuzzy logic

L. A. Zadeh

Why?

- Omnipresent in natural language (people need to evaluate); occur in description of any process, decision situation, procedure, characterization of objects, etc.
- Belong to the agenda of fuzzy logic
  Fuzzy IF-THEN rules, Linguistic variable
- They are present in applications of fuzzy logic
  Fuzzy control, decision making, classification, various industrial applications

Example of algorithm used by the geologist

1. Ends of sequences are usually rocks types 6, 7, or 8, if they are followed by rock type 1,2, or 3. If the given rock has lower number followed again by 6, 7, or 8 and it is *too thin* then it is ignored.
2. Check whether the obtained sequences are *sufficiently thick*. If the given sequence is *too thin* then it is joined with the following one, provided that the resulting sequence does not become *too thick*.
3. If the sequence is *too thick* then it is further divided: check all rock types 4 and mark them as ends of new sequence provided that the new sequence is not *too thin*; mark the new sequence only if it is *sufficiently thick*.

Grammatical structure

(TEv-expressions)

(i) Simple evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ := ⟨linguistic hedge⟩⟨TE-adjective⟩

⟨TE-adjectives⟩: gradable adjectives, adjectives of manner, and possibly some other ones

⟨linguistic hedge⟩ — intensifying adverb

⟨linguistic hedge⟩ := ∅ | ⟨narrowing adverb⟩ | ⟨widening adverb⟩ | ⟨specifying adverb⟩

(b) ⟨fuzzy quantity⟩ := ⟨linguistic hedge⟩⟨numeral⟩

⟨numeral⟩ — name of element from the considered scale

Grammatical structure

(TEv-expressions)

(i) Simple evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ := ⟨linguistic hedge⟩⟨TE-adjective⟩

⟨TE-adjectives⟩: gradable adjectives, adjectives of manner, and possibly some other ones

⟨linguistic hedge⟩ — intensifying adverb

⟨linguistic hedge⟩ := ∅ | ⟨narrowing adverb⟩ | ⟨widening adverb⟩ | ⟨specifying adverb⟩

(b) ⟨fuzzy quantity⟩ := ⟨linguistic hedge⟩⟨numeral⟩

⟨numeral⟩ — name of element from the considered scale

Grammatical structure

(TEv-expressions)

(i) Simple evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ := ⟨linguistic hedge⟩⟨TE-adjective⟩

⟨TE-adjectives⟩: gradable adjectives, adjectives of manner, and possibly some other ones

⟨linguistic hedge⟩ — intensifying adverb

⟨linguistic hedge⟩ := $\emptyset$ | ⟨narrowing adverb⟩ | ⟨widening adverb⟩ | ⟨specifying adverb⟩

(b) ⟨fuzzy quantity⟩ := ⟨linguistic hedge⟩⟨numeral⟩

⟨numeral⟩ — name of element from the considered scale

Grammatical structure

(TEv-expressions)

(i) Simple evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ := ⟨linguistic hedge⟩⟨TE-adjective⟩

⟨TE-adjectives⟩: gradable adjectives, adjectives of manner, and possibly some other ones

⟨linguistic hedge⟩ — intensifying adverb

⟨linguistic hedge⟩ := $\emptyset$ | ⟨narrowing adverb⟩ | ⟨widening adverb⟩ | ⟨specifying adverb⟩

(b) ⟨fuzzy quantity⟩ := ⟨linguistic hedge⟩⟨numeral⟩
⟨numeral⟩ — name of element from the considered scale

Grammatical structure

(TEv-expressions)

(i) Simple evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ := ⟨linguistic hedge⟩⟨TE-adjective⟩

⟨TE-adjectives⟩: gradable adjectives, adjectives of manner, and possibly some other ones

⟨linguistic hedge⟩ — intensifying adverb

⟨linguistic hedge⟩ := $\emptyset$ | ⟨narrowing adverb⟩ | ⟨widening adverb⟩ | ⟨specifying adverb⟩

(b) ⟨fuzzy quantity⟩ := ⟨linguistic hedge⟩⟨numeral⟩

⟨numeral⟩ — name of element from the considered scale

Grammatical structure

(ii) Negative evaluative expression:

not ⟨trichotomous evaluative expression⟩

(iii) Compound evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ or ⟨trichotomous evaluative expression⟩

(b) ⟨trichotomous evaluative expression⟩ and/but ⟨negative evaluative expression⟩

Grammatical structure

(ii) Negative evaluative expression:

not ⟨trichotomous evaluative expression⟩

(iii) Compound evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ or ⟨trichotomous evaluative expression⟩

(b) ⟨trichotomous evaluative expression⟩ and/but ⟨negative evaluative expression⟩

Grammatical structure

(ii) Negative evaluative expression:

not ⟨trichotomous evaluative expression⟩

(iii) Compound evaluative expression:

(a) ⟨trichotomous evaluative expression⟩ or ⟨trichotomous evaluative expression⟩

(b) ⟨trichotomous evaluative expression⟩ and/but ⟨negative evaluative expression⟩

Grammatical structure

(ii) Negative evaluative expression:

not ⟨trichotomous evaluative expression⟩

(iii) Compound evaluative expression:

- **(a)** ⟨trichotomous evaluative expression⟩ or ⟨trichotomous evaluative expression⟩
- **(b)** ⟨trichotomous evaluative expression⟩ and/but ⟨negative evaluative expression⟩

TE-adjectives

Antonyms

$$\langle\text{TE-adjective}\rangle \longleftrightarrow \langle\text{antonym}\rangle$$

young ⟷ *old; ugly* ⟷ *nice; stupid* ⟷ *clever; excellent* ⟷ *poor*

Fundamental evaluative trichotomy

$$\langle\text{TE-adjective}\rangle \text{ —} \langle\text{middle member}\rangle \text{ —} \langle\text{antonym}\rangle$$

young — medium age — old; ugly — normal — nice; stupid — medium intelligent — clever

TE-adjectives

Antonyms

$$\langle \text{TE-adjective} \rangle \longleftrightarrow \langle \text{antonym} \rangle$$

young $\longleftrightarrow$ *old; ugly* $\longleftrightarrow$ *nice; stupid* $\longleftrightarrow$ *clever; excellent* $\longleftrightarrow$ *poor*

Fundamental evaluative trichotomy

$$\langle \text{TE-adjective} \rangle \text{ — } \langle \text{middle member} \rangle \text{ — } \langle \text{antonym} \rangle$$

young — medium age — old; ugly — normal — nice; stupid — medium intelligent — clever

Grammatical structure

Example

TE-adjectives: **small, medium, big** – canonical

weak, medium strong, strong; silly, normal, intelligent

Fuzzy numbers: *twenty five, roughly one hundred*

Simple evaluative expressions: *very short, more or less strong, more or less medium, roughly big, about twenty five*

Negative evaluative expressions: *not short, not very deep*

Compound evaluative expressions: *roughly small or medium, small but not very (small)*

Evaluative linguistic predications

Let $\mathcal{A}$ — evaluative linguistic expression

(a) Evaluative (linguistic) predication

$$\langle\text{noun}\rangle \text{ is } \mathcal{A}$$

(b) Abstracted evaluative predication

$$X \text{ is } \mathcal{A}$$

X — variable, values are arbitrary elements

(c) Compound evaluative predications:

$$\mathcal{A} \text{ and } \mathcal{B} \qquad \mathcal{A} \text{ or } \mathcal{B}$$

(d) Fuzzy IF-THEN rule — abstracted conditional clause

$$\mathcal{R} := \text{ IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}.$$

IRAFM

Evaluative linguistic predications

Let $\mathcal{A}$ — evaluative linguistic expression

(a) Evaluative (linguistic) predication

$$\langle\text{noun}\rangle \text{ is } \mathcal{A}$$

(b) Abstracted evaluative predication

$$X \text{ is } \mathcal{A}$$

X — variable, values are arbitrary elements

(c) Compound evaluative predications:

$$\mathcal{A} \text{ and } \mathcal{B} \qquad \mathcal{A} \text{ or } \mathcal{B}$$

(d) Fuzzy IF-THEN rule — abstracted conditional clause

$$\mathcal{R} := \text{IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}.$$

Evaluative linguistic predications

Let $\mathcal{A}$ — evaluative linguistic expression

(a) Evaluative (linguistic) predication

$$\langle\text{noun}\rangle \text{ is } \mathcal{A}$$

(b) Abstracted evaluative predication

$$X \text{ is } \mathcal{A}$$

X — variable, values are arbitrary elements

(c) Compound evaluative predications:

$$\mathcal{A} \text{ and } \mathcal{B} \qquad \mathcal{A} \text{ or } \mathcal{B}$$

(d) Fuzzy IF-THEN rule — abstracted conditional clause

$$\mathcal{R} := \text{IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}.$$

Evaluative linguistic predications

Let $\mathcal{A}$ — evaluative linguistic expression

(a) Evaluative (linguistic) predication

$$\langle\text{noun}\rangle \text{ is } \mathcal{A}$$

(b) Abstracted evaluative predication

$$X \text{ is } \mathcal{A}$$

X — variable, values are arbitrary elements

(c) Compound evaluative predications:

$$\mathcal{A} \text{ and } \mathcal{B} \qquad \mathcal{A} \text{ or } \mathcal{B}$$

(d) Fuzzy IF-THEN rule — abstracted conditional clause

$$\mathcal{R} := \text{IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}.$$

Evaluative linguistic predications

Let $\mathcal{A}$ — evaluative linguistic expression

(a) Evaluative (linguistic) predication

$$\langle\text{noun}\rangle \text{ is } \mathcal{A}$$

(b) Abstracted evaluative predication

$$X \text{ is } \mathcal{A}$$

X — variable, values are arbitrary elements

(c) Compound evaluative predications:

$$\mathcal{A} \text{ and } \mathcal{B} \qquad \mathcal{A} \text{ or } \mathcal{B}$$

(d) Fuzzy IF-THEN rule — abstracted conditional clause

$$\mathcal{R} := \text{ IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}.$$

Evaluative linguistic predications

Evaluative predication — general relationship

$$\mathcal{A}\ \langle\text{noun}\rangle,$$

small house $\equiv$ *house is small*
very tall man $\equiv$ *a man is very tall*

Objects named by $\langle$noun$\rangle$ — quite complicated entities

$\mathcal{A}$ concerns certain *feature* (or few features) of objects (attain values from some ordered scale)
Values are evaluated by $\mathcal{A}$

IRAFM

Evaluative linguistic predications

Evaluative predication — general relationship

$$\mathcal{A}\ \langle\text{noun}\rangle,$$

small house $\equiv$ *house is small*
very tall man $\equiv$ *a man is very tall*

Objects named by $\langle$noun$\rangle$ — quite complicated entities

$\mathcal{A}$ concerns certain *feature* (or few features) of objects (attain values from some ordered scale)
Values are evaluated by $\mathcal{A}$

IRAFM

Evaluative linguistic predications

Evaluative predication — general relationship

$$\mathcal{A}\ \langle\text{noun}\rangle,$$

small house $\equiv$ *house is small*
very tall man $\equiv$ *a man is very tall*

Objects named by $\langle$noun$\rangle$ — quite complicated entities

$\mathcal{A}$ concerns certain *feature* (or few features) of objects (attain values from some ordered scale)
Values are evaluated by $\mathcal{A}$

IRAFM

Informal principles of the meaning of TEv-expressions

Any model of the meaning of evaluative linguistic expressions must capture the following concepts:

- Context (possible world) — a state of the world at given time moment and place
- Intension — a property; it may lead to different truth values in various contexts; (*invariant with respect to various contexts*)
- Extension — a class of elements determined by an intension in a given context; (*it does change when changing the context*)
- *Vagueness of the meaning of natural language expressions is a consequence of the indiscernibility between objects*

IRAFM

Informal principles of the meaning of TEv-expressions

Any model of the meaning of evaluative linguistic expressions must capture the following concepts:

- Context (possible world) — a state of the world at given time moment and place
- Intension — a property; it may lead to different truth values in various contexts; (*invariant with respect to various contexts*)
- Extension — a class of elements determined by an intension in a given context; (*it does change when changing the context*)
- *Vagueness of the meaning of natural language expressions is a consequence of the indiscernibility between objects*

IRAFM

Informal principles of the meaning of TEv-expressions

Any model of the meaning of evaluative linguistic expressions must capture the following concepts:

- Context (possible world) — a state of the world at given time moment and place
- Intension — a property; it may lead to different truth values in various contexts; (*invariant with respect to various contexts*)
- Extension — a class of elements determined by an intension in a given context; (*it does change when changing the context*)
- *Vagueness of the meaning of natural language expressions is a consequence of the indiscernibility between objects*

Informal principles of the meaning of TEv-expressions

Any model of the meaning of evaluative linguistic expressions must capture the following concepts:

- Context (possible world) — a state of the world at given time moment and place
- Intension — a property; it may lead to different truth values in various contexts; (*invariant with respect to various contexts*)
- Extension — a class of elements determined by an intension in a given context; (*it does change when changing the context*)
- *Vagueness of the meaning of natural language expressions is a consequence of the indiscernibility between objects*

Informal principles of the meaning of TEv-expressions

Any model of the meaning of evaluative linguistic expressions must capture the following concepts:

- Context (possible world) — a state of the world at given time moment and place
- Intension — a property; it may lead to different truth values in various contexts; (*invariant with respect to various contexts*)
- Extension — a class of elements determined by an intension in a given context; (*it does change when changing the context*)
- *Vagueness of the meaning of natural language expressions is a consequence of the indiscernibility between objects*

Informal principles of the meaning of TEv-expressions

(A) *Linguistic context* — nonempty, linearly ordered and bounded scale

Three distinguished limit points: *left bound*, *right bound*, and a *central point*

(B) *Intension* — function from the set of contexts into a set of fuzzy sets.
Each context is assigned a fuzzy set inside it — *extension* of TEv-expression in the given context.

IRAFM

Informal principles of the meaning of TEv-expressions

(A) *Linguistic context* — nonempty, linearly ordered and bounded scale

Three distinguished limit points: *left bound*, *right bound*, and a *central point*

(B) *Intension* — function from the set of contexts into a set of fuzzy sets.
Each context is assigned a fuzzy set inside it — *extension* of TEv-expression in the given context.

IRAFM

Informal principles of the meaning of TEv-expressions

(C) Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)

Three horizons in each context:

(a) a horizon from the left bound towards central point,
(b) a horizon from the right bound back towards central point,
(c) a horizon from the central point towards both left and right bounds

(D) Each horizon is represented by a fuzzy set determined by a reasoning analogous to that leading to the sorites paradox

Informal principles of the meaning of TEv-expressions

(C) Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)
Three horizons in each context:

- **(a)** a horizon from the left bound towards central point,
- **(b)** a horizon from the right bound back towards central point,
- **(c)** a horizon from the central point towards both left and right bounds

(D) Each horizon is represented by a fuzzy set determined by a reasoning analogous to that leading to the sorites paradox

Informal principles of the meaning of TEv-expressions

(C) Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)
Three horizons in each context:

- **(a)** a horizon from the left bound towards central point,
- **(b)** a horizon from the right bound back towards central point,
- **(c)** a horizon from the central point towards both left and right bounds

(D) Each horizon is represented by a fuzzy set determined by a reasoning analogous to that leading to the sorites paradox

Informal principles of the meaning of TEv-expressions

(E) *Extension* of each TEv-expression is delineated by a specific horizon obtained by *modification of the horizon*: linguistic hedge

Shifting the horizon — moving it closer to, or farther from the limit point (decreasing the truth values)

Small decrease for big truth values; big for small ones
Limit points — *typical values* of TEv-expression

IRAFM

Informal principles of the meaning of TEv-expressions

(E) *Extension* of each TEv-expression is delineated by a specific horizon obtained by *modification of the horizon*: linguistic hedge

Shifting the horizon — moving it closer to, or farther from the limit point (decreasing the truth values)

Small decrease for big truth values; big for small ones
Limit points — *typical values* of TEv-expression

IRAFM

Informal principles of the meaning of TEv-expressions

(E) *Extension* of each TEv-expression is delineated by a specific horizon obtained by *modification of the horizon*: linguistic hedge

Shifting the horizon — moving it closer to, or farther from the limit point (decreasing the truth values)

Small decrease for big truth values; big for small ones

Limit points — *typical values* of TEv-expression

Informal principles of the meaning of TEv-expressions

(E) *Extension* of each TEv-expression is delineated by a specific horizon obtained by *modification of the horizon*: linguistic hedge

Shifting the horizon — moving it closer to, or farther from the limit point (decreasing the truth values)

Small decrease for big truth values; big for small ones

Limit points — *typical values* of TEv-expression

Informal principles of the meaning of TEv-expressions

(E) *Extension* of each TEv-expression is delineated by a specific horizon obtained by *modification of the horizon*: linguistic hedge

Shifting the horizon — moving it closer to, or farther from the limit point (decreasing the truth values)

Small decrease for big truth values; big for small ones
Limit points — *typical values* of TEv-expression

Informal principles of the meaning of TEv-expressions

(F) Each scale is vaguely partitioned by the fundamental evaluative trichotomy

No element of the context falls into extensions of both antonyms

Any element of the scale is contained in the extension of at most two neighboring expressions from the fundamental evaluative trichotomy

Informal principles of the meaning of TEv-expressions

(F) Each scale is vaguely partitioned by the fundamental evaluative trichotomy

No element of the context falls into extensions of both antonyms

Any element of the scale is contained in the extension of at most two neighboring expressions from the fundamental evaluative trichotomy

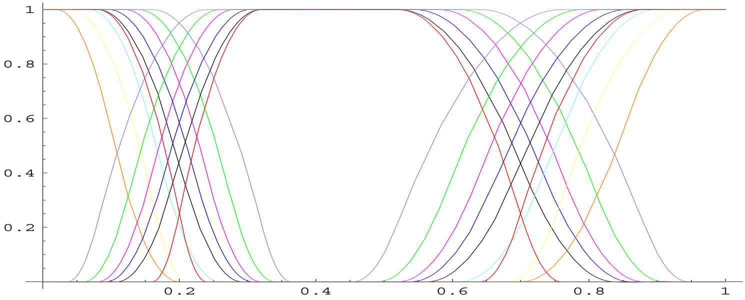
IRAFM

Informal principles of the meaning of TEv-expressions

(F) Each scale is vaguely partitioned by the fundamental evaluative trichotomy

No element of the context falls into extensions of both antonyms
Any element of the scale is contained in the extension of at most two neighboring expressions from the fundamental evaluative trichotomy

Extensions of evaluative expressions



Fuzzy type theory — syntax

Types: o (truth values), ϵ (elements)

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\beta \equiv B_\beta$, $\lambda x_\alpha\, C_\beta$, Δ_{oo}

Interpretation of formulas $A_{\beta\alpha}$: functions $M_\alpha \longrightarrow M_\beta$

Formulas of type o are propositions
Interpretation of $A_{o\alpha}$ is a fuzzy subset of M_α

Description operator $\iota_{\alpha(o\alpha)}$

Fuzzy type theory — syntax

Types: o (truth values), ϵ (elements)

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\beta \equiv B_\beta$, $\lambda x_\alpha\, C_\beta$, $\mathbf{\Delta}_{oo}$

Interpretation of formulas $A_{\beta\alpha}$: functions $M_\alpha \longrightarrow M_\beta$

Formulas of type o are propositions
Interpretation of $A_{o\alpha}$ is a fuzzy subset of M_α

Description operator $\iota_{\alpha(o\alpha)}$

Fuzzy type theory — syntax

Types: o (truth values), ϵ (elements)

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\beta \equiv B_\beta$, $\lambda x_\alpha\, C_\beta$, $\mathbf{\Delta}_{oo}$

Interpretation of formulas $A_{\beta\alpha}$: functions $M_\alpha \longrightarrow M_\beta$

Formulas of type o are propositions
Interpretation of $A_{o\alpha}$ is a fuzzy subset of M_α

Description operator $\iota_{\alpha(o\alpha)}$

Fuzzy type theory — syntax

Types: o (truth values), ϵ (elements)

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\beta \equiv B_\beta$, $\lambda x_\alpha\, C_\beta$, $\mathbf{\Delta}_{oo}$

Interpretation of formulas $A_{\beta\alpha}$: functions $M_\alpha \longrightarrow M_\beta$

Formulas of type o are propositions
Interpretation of $A_{o\alpha}$ is a fuzzy subset of M_α

Description operator $\iota_{\alpha(o\alpha)}$

Fuzzy type theory — syntax

Types: o (truth values), ϵ (elements)

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\beta \equiv B_\beta$, $\lambda x_\alpha\, C_\beta$, $\mathbf{\Delta}_{oo}$

Interpretation of formulas $A_{\beta\alpha}$: functions $M_\alpha \longrightarrow M_\beta$

Formulas of type o are propositions
Interpretation of $A_{o\alpha}$ is a fuzzy subset of M_α

Description operator $\iota_{\alpha(o\alpha)}$

Fuzzy type theory — semantics I

Frame $\mathcal{M}$

$$\mathcal{I} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$$

(i) $\mathcal{L}_\Delta$: IMTL$_\Delta$-algebra or Łukasiewicz Δ-algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, \otimes, \Delta, \rightarrow, 0, 1 \rangle$$

$$\begin{aligned} \vee, \wedge &= \text{minimum, maximum} \\ \otimes &= \text{left continuous t-norm}, \qquad a \otimes b = 0 \vee (a + b - 1) \\ \rightarrow &= \text{residuation} \qquad a \rightarrow b = 1 \wedge (1 - a + b) \\ \neg a = a \rightarrow 0 &= 1 - a, \qquad \neg\neg a = a \end{aligned}$$

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Fuzzy type theory — semantics I

Frame $\mathcal{M}$

$$\mathcal{I} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L}_\Delta \rangle$$

(i) $\mathcal{L}_\Delta$: IMTL$_\Delta$-algebra or Łukasiewicz Δ-algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, \otimes, \Delta, \rightarrow, 0, 1 \rangle$$

$$\vee, \wedge = \text{minimum, maximum}$$

$$\otimes = \text{left continuous t-norm}, \qquad a \otimes b = 0 \vee (a + b - 1)$$

$$\rightarrow = \text{residuation} \qquad a \rightarrow b = 1 \wedge (1 - a + b)$$

$$\neg a = a \rightarrow 0 = 1 - a, \qquad \neg\neg a = a$$

$$\Delta(a) = \begin{cases} \mathbf{1} & \text{if } a = \mathbf{1}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Fuzzy type theory — semantics II

(ii) Fuzzy equality $=_\alpha : M_\alpha \times M_\alpha \longrightarrow L$

$$[x =_\alpha x] = \mathbf{1} \qquad \text{(reflexivity)}$$
$$[x =_\alpha y] = [y =_\alpha x] \qquad \text{(symmetry)}$$
$$[x =_\alpha y] \otimes [y =_\alpha z] \leq [x =_\alpha z] \qquad \text{(transitivity)}$$

Example

$$[x =_\alpha y] = 0 \vee (1 - |x - y|)$$

$$[x =_\alpha y] = \begin{cases} 1, \text{if } x = y \\ \frac{1}{v-u} \cdot ((v - x) \wedge (v - y)) \vee ((x - u) \wedge (y - u)) \end{cases}$$

Fuzzy type theory — semantics II

(ii) Fuzzy equality $=_\alpha\colon M_\alpha \times M_\alpha \longrightarrow L$

$$[x =_\alpha x] = \mathbf{1} \qquad \text{(reflexivity)}$$

$$[x =_\alpha y] = [y =_\alpha x] \qquad \text{(symmetry)}$$

$$[x =_\alpha y] \otimes [y =_\alpha z] \leq [x =_\alpha z] \qquad \text{(transitivity)}$$

Example

$$[x =_\alpha y] = 0 \vee (1 - |x - y|)$$

$$[x =_\alpha y] = \begin{cases} 1, \text{if } x = y \\ \frac{1}{v-u} \cdot ((v - x) \wedge (v - y)) \vee ((x - u) \wedge (y - u)) \end{cases}$$

Scheme of frame

$$\vdots$$

....

$$(M_{\beta\alpha} \subseteq \{f_{\beta\alpha} \mid f_{\beta\alpha} : M_\alpha \longrightarrow M_\beta\}, =_{\beta\alpha})$$

....

$$\vdots$$

....

$$(M_{oo} \subseteq \{g_{oo} \mid g_{oo} : M_o \longrightarrow M_o\}, =_{oo})$$

$$(M_{o\epsilon} \subseteq \{f_{o\epsilon} \mid f_{o\epsilon} : M_\epsilon \longrightarrow M_o\}, =_{o\epsilon})$$

$$(M_{\epsilon\epsilon} \subseteq \{f_{\epsilon\epsilon} \mid f_{\epsilon\epsilon} : M_\epsilon \longrightarrow M_\epsilon\}, =_{\epsilon\epsilon}), \ldots$$

....

$$(M_o = \{a \mid a \in L\}, \leftrightarrow) \qquad (M_\epsilon = \{u \mid \varphi(u)\}, =_\epsilon)$$

Interpretation

$$\mathcal{I}^{\mathcal{M}}(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}}$$

Examples of interpretation:

- $\mathcal{I}^{\mathcal{M}}(A_o) \in L$ – a truth value
- $\mathcal{I}^{\mathcal{M}}(A_{o\epsilon})$ — fuzzy set in M_ϵ
- $\mathcal{I}^{\mathcal{M}}(\iota_{\alpha(o\alpha)})$ is a defuzzification operation
- $\mathcal{I}^{\mathcal{M}}(\Delta A_o) \in \{\mathbf{0}, \mathbf{1}\}$ – crisp truth value

IRAFM

Interpretation

$$\mathcal{I}^{\mathcal{M}}(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}}$$

Examples of interpretation:

- $\mathcal{I}^{\mathcal{M}}(A_o) \in L$ – a truth value
- $\mathcal{I}^{\mathcal{M}}(A_{o\epsilon})$ — fuzzy set in M_ϵ
- $\mathcal{I}^{\mathcal{M}}(\iota_{\alpha(o\alpha)})$ is a defuzzification operation
- $\mathcal{I}^{\mathcal{M}}(\Delta A_o) \in \{\mathbf{0}, \mathbf{1}\}$ – crisp truth value

Interpretation

$$\mathcal{I}^{\mathcal{M}}(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}}$$

Examples of interpretation:

- $\mathcal{I}^{\mathcal{M}}(A_o) \in L$ – a truth value
- $\mathcal{I}^{\mathcal{M}}(A_{o\epsilon})$ — fuzzy set in M_ϵ
- $\mathcal{I}^{\mathcal{M}}(\iota_{\alpha(o\alpha)})$ is a defuzzification operation
- $\mathcal{I}^{\mathcal{M}}(\Delta A_o) \in \{0, 1\}$ – crisp truth value

Interpretation

$$\mathcal{I}^{\mathcal{M}}(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}}$$

Examples of interpretation:

- $\mathcal{I}^{\mathcal{M}}(A_o) \in L$ – a truth value
- $\mathcal{I}^{\mathcal{M}}(A_{o\epsilon})$ — fuzzy set in M_ϵ
- $\mathcal{I}^{\mathcal{M}}(\iota_{\alpha(o\alpha)})$ is a defuzzification operation
- $\mathcal{I}^{\mathcal{M}}(\Delta A_o) \in \{0, 1\}$ – crisp truth value

IRAFM

Interpretation

$$\mathcal{I}^{\mathcal{M}}(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}}$$

Examples of interpretation:

- $\mathcal{I}^{\mathcal{M}}(A_o) \in L$ – a truth value
- $\mathcal{I}^{\mathcal{M}}(A_{o\epsilon})$ — fuzzy set in M_ϵ
- $\mathcal{I}^{\mathcal{M}}(\iota_{\alpha(o\alpha)})$ is a defuzzification operation
- $\mathcal{I}^{\mathcal{M}}(\boldsymbol{\Delta} A_o) \in \{\mathbf{0}, \mathbf{1}\}$ – crisp truth value

Formal theory of TEv-expressions

Construction of formal theory T^{Ev} in the language of FTT formalizing general characteristics (A)–(F)

All properties are consequences of 11 special axioms of T^{Ev}

Formal syntactical proofs of all properties!

Example (special axioms)

(EV7) $\Delta((t \Rightarrow u) \& (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u,$

(EV8) $t \equiv t' \,\&\, z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z',$

Formal theory of TEv-expressions

Construction of formal theory T^{Ev} in the language of FTT formalizing general characteristics (A)–(F)

All properties are consequences of 11 special axioms of T^{Ev}

Formal syntactical proofs of all properties!

Example (special axioms)

(EV7) $\Delta((t \Rightarrow u) \& (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u,$

(EV8) $t \equiv t' \,\&\, z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z',$

IRAFM

Formal theory of TEv-expressions

Construction of formal theory T^{Ev} in the language of FTT formalizing general characteristics (A)–(F)

All properties are consequences of 11 special axioms of T^{Ev}

Formal syntactical proofs of all properties!

Example (special axioms)

(EV7) $\Delta((t \Rightarrow u) \,\&\, (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u,$

(EV8) $t \equiv t' \,\&\, z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z',$

Context

(A) Nonempty, linearly ordered and bounded scale, three distinguished limit points: *left bound*, *right bound*, and a *central point*

Context $w : [0, 1] \longrightarrow M$:

$$w(0) = v_L \qquad \text{(left bound)}$$
$$w(0.5) = v_S \qquad \text{(central point)}$$
$$w(1) = v_R \qquad \text{(right bound)}$$

Set of contexts $W = \{w \mid w : [0, 1] \longrightarrow M\}$
$w^{-1} := \imath x \cdot y \equiv wx$
Linear ordering in each context w

$$y \leq_w y' \quad \text{iff} \quad w^{-1}y \Rightarrow w^{-1}y', \qquad y, y' \in w$$

IRAFM

Context

(A) Nonempty, linearly ordered and bounded scale, three distinguished limit points: *left bound*, *right bound*, and a *central point*

Context $w : [0,1] \longrightarrow M$:

$$w(0) = v_L \quad \text{(left bound)}$$
$$w(0.5) = v_S \quad \text{(central point)}$$
$$w(1) = v_R \quad \text{(right bound)}$$

Set of contexts $W = \{w \mid w : [0,1] \longrightarrow M\}$
$w^{-1} := \imath x \cdot y \equiv wx$
Linear ordering in each context w

$$y \leq_w y' \quad \text{iff} \quad w^{-1}y \Rightarrow w^{-1}y', \qquad y, y' \in w$$

Intension

(B) Function from the set of contexts into a set of fuzzy sets

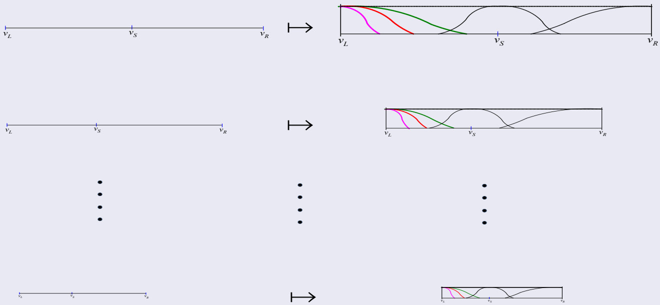
Scheme of intension

Intension

(B) Function from the set of contexts into a set of fuzzy sets

Scheme of intension

$$W \longrightarrow \mathcal{F}(w\ ([0,1]))$$



Horizon

(C) Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)

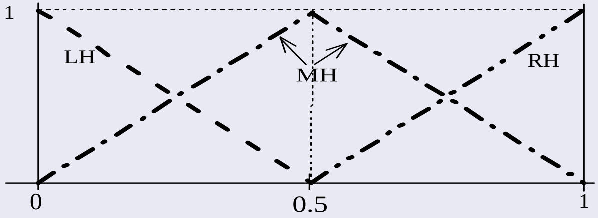
Three horizons

IRAFM

Horizon

(C) Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)

Three horizons



Special fuzzy equality

Fuzzy equality $\sim$ on the set $[0, 1]$ of truth values:

- $[0 \sim 0.5] = 0$
- if $a \leq b \leq c$ then $[a \sim c] \leq [a \sim b]$
- there is a such that $0 < [0 \sim a] < 1$

Example

Standard Łukasiewicz MV-algebra of truth values:

$$[a \sim b] = \frac{0.5 - |a - b|}{0.5}.$$

Horizon

Fuzzy equality induced by $\sim$

$$y \approx_w y' := \quad \text{iff} \quad w^{-1}y \sim w^{-1}y', \quad x, y \in w$$

Three horizons

$$\begin{aligned} LH(a) &= [0 \sim a], & LH(w\,x) &= [v_L \approx_w x] \\ MH(a) &= [0.5 \sim a], & MH(w\,x) &= [v_S \approx_w x] \\ RH(a) &= [1 \sim a], & RH(w\,x) &= [v_R \approx_w x] \end{aligned}$$

Properties of horizon

(D) Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

Special context w_N: $v_L = 0$, $v_S = p$, $v_R = q$

$$\mathbb{FN}(n) = [0 \approx_{w_N} n] \text{ — finite numbers do not form a heap}$$

Theorem

(a) $\mathbb{FN}(0) = 1$

(b) $p \leq n \Rightarrow \mathbb{FN}(n) = 0$

(c) $\mathbb{FN}(n + 1) \leq \mathbb{FN}(n)$

(d) *No n such that* $\mathbb{FN}(n) = 1$ *and* $\mathbb{FN}(n + 1) = 0$

Properties of horizon

(D) Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

Special context w_N: $v_L = 0$, $v_S = p$, $v_R = q$

$$\mathbb{FN}(n) = [0 \approx_{w_N} n] \text{ — finite numbers do not form a heap}$$

Theorem

(a) $\mathbb{FN}(0) = 1$

(b) $p \leq n \Rightarrow \mathbb{FN}(n) = 0$

(c) $\mathbb{FN}(n+1) \leq \mathbb{FN}(n)$

(d) *No n such that* $\mathbb{FN}(n) = 1$ *and* $\mathbb{FN}(n+1) = 0$

Properties of horizon

(D) Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

Special context w_N: $v_L = 0$, $v_S = p$, $v_R = q$

$$\mathbb{FN}(n) = [0 \approx_{w_N} n]$$ — finite numbers do not form a heap

Theorem

(a) $\mathbb{FN}(0) = 1$

(b) $p \leq n \Rightarrow \mathbb{FN}(n) = 0$

(c) $\mathbb{FN}(n + 1) \leq \mathbb{FN}(n)$

(d) *No n such that* $\mathbb{FN}(n) = 1$ *and* $\mathbb{FN}(n + 1) = 0$

Hedge: shift of the horizon

(E) *Extension* of each TEv-expression is delineated by *modification (shifting) of the horizon*
Modification: linguistic hedge

Hedges — horizon modifications

$$\nu : [0, 1] \longrightarrow [0, 1]$$

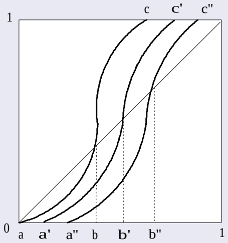
IRAFM

Hedge: shift of the horizon

(E) *Extension* of each TEv-expression is delineated by *modification (shifting) of the horizon*
Modification: linguistic hedge

Hedges — horizon modifications

$$\nu : [0, 1] \longrightarrow [0, 1]$$



Formalization of intension and extension

Intension — a function $W \longrightarrow \mathcal{F}(w([0,1]))$

(i) *S-intension:*

$$Sm\,\nu = \lambda w\,\lambda x \cdot \boldsymbol{\nu}(LH(w^{-1}x))$$

(ii) *M-intension:*

$$Me\,\nu = \lambda w\,\lambda x \cdot \boldsymbol{\nu}(MH(w^{-1}x))$$

(iii) *B-intension:*

$$Bi\,\nu = \lambda w\,\lambda x \cdot \boldsymbol{\nu}(RH(w^{-1}x))$$

Extension of an evaluative predication in the context w

$$(Ev\,\nu)w \underset{\sim}{\subset} w([0,1])$$

Formalization of intension and extension

Intension — a function $W \longrightarrow \mathcal{F}(w([0,1]))$

(i) *S-intension:*

$$Sm\,\nu = \lambda w\,\lambda x \cdot \boldsymbol{\nu}(LH(w^{-1}x))$$

(ii) *M-intension:*

$$Me\,\nu = \lambda w\,\lambda x \cdot \boldsymbol{\nu}(MH(w^{-1}x))$$

(iii) *B-intension:*

$$Bi\,\nu = \lambda w\,\lambda x \cdot \boldsymbol{\nu}(RH(w^{-1}x))$$

Extension of an evaluative predication in the context w

$$(Ev\,\nu)w \underset{\sim}{\subset} w([0,1])$$

Fundamental evaluative trichotomy

(F) Each scale is vaguely partitioned by the fundamental evaluative trichotomy

Special hedge $\bar{\nu}$

$$\overline{Sm}(w) = \lambda x \cdot \bar{\nu}(LH(w^{-1}x))$$
$$\overline{Me}(w) = \lambda x \cdot \bar{\nu}(MH(w^{-1}x))$$
$$\overline{Bi}(w) = \lambda x \cdot \bar{\nu}(RH(w^{-1}x))$$

IRAFM

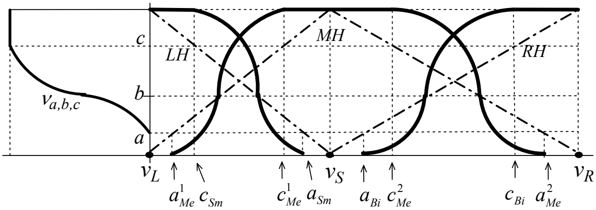
Fundamental evaluative trichotomy

(F) Each scale is vaguely partitioned by the fundamental evaluative trichotomy

Special hedge $\bar{\nu}$

$$\overline{Sm}(w) = \lambda x \cdot \bar{\boldsymbol{\nu}}(LH(w^{-1}x))$$
$$\overline{Me}(w) = \lambda x \cdot \bar{\boldsymbol{\nu}}(MH(w^{-1}x))$$
$$\overline{Bi}(w) = \lambda x \cdot \bar{\boldsymbol{\nu}}(RH(w^{-1}x))$$

Fundamental evaluative trichotomy



Intensions of evaluative predications

$\langle$linguistic hedge$\rangle \mapsto \boldsymbol{\nu}$

- $\mathrm{Int}(X$ is $\langle$linguistic hedge$\rangle$ small$) := Sm\,\boldsymbol{\nu}$
- $\mathrm{Int}(X$ is $\langle$linguistic hedge$\rangle$ medium$) := Me\,\boldsymbol{\nu}$
- $\mathrm{Int}(X$ is $\langle$linguistic hedge$\rangle$ big$) := Bi\,\boldsymbol{\nu}$

Special case:

$$\mathrm{Int}(X \text{ is small}) := \overline{Sm}$$
$$\mathrm{Int}(X \text{ is medium}) := \overline{Me}$$
$$\mathrm{Int}(X \text{ is big}) := \overline{Bi}$$

Theorem

Formal theory T^{Ev} of evaluative linguistic expressions is consistent

Intensions of evaluative predications

$\langle$linguistic hedge$\rangle \mapsto \boldsymbol{\nu}$

- $\mathrm{Int}(X$ is $\langle$linguistic hedge$\rangle$ small$) := \ Sm\,\boldsymbol{\nu}$
- $\mathrm{Int}(X$ is $\langle$linguistic hedge$\rangle$ medium$) := \ Me\,\boldsymbol{\nu}$
- $\mathrm{Int}(X$ is $\langle$linguistic hedge$\rangle$ big$) := \ Bi\,\boldsymbol{\nu}$

Special case:

$$\mathrm{Int}(X \text{ is small}) := \overline{Sm}$$
$$\mathrm{Int}(X \text{ is medium}) := \overline{Me}$$
$$\mathrm{Int}(X \text{ is big}) := \overline{Bi}$$

Theorem

Formal theory T^{Ev} of evaluative linguistic expressions is consistent

Conclusions

- Development of a comprehensive logical theory of evaluative linguistic expressions; trichotomous evaluative expressions
- We can distinguish the meaning of simple expressions like *medium, very small, extremely big*, etc., from the meaning of evaluative predications
- Our theory can be further extended to cover fuzzy quantities, negative and compound evaluative expressions
- Our theory can be further extended to intermediate (generalized) quantifiers (*most, a lot of, at least, at most, few*, etc. Generalized Aristotle's syllogisms.

IRAFM

Conclusions

- Development of a comprehensive logical theory of evaluative linguistic expressions; trichotomous evaluative expressions
- We can distinguish the meaning of simple expressions like *medium, very small, extremely big*, etc., from the meaning of evaluative predications
- Our theory can be further extended to cover fuzzy quantities, negative and compound evaluative expressions
- Our theory can be further extended to intermediate (generalized) quantifiers (*most, a lot of, at least, at most, few*, etc. Generalized Aristotle's syllogisms.

IRAFM

Conclusions

- Development of a comprehensive logical theory of evaluative linguistic expressions;
  trichotomous evaluative expressions
- We can distinguish the meaning of simple expressions like *medium, very small, extremely big*, etc., from the meaning of evaluative predications
- Our theory can be further extended to cover fuzzy quantities, negative and compound evaluative expressions
- Our theory can be further extended to intermediate (generalized) quantifiers (*most, a lot of, at least, at most, few*, etc. Generalized Aristotle's syllogisms.

Conclusions

- Development of a comprehensive logical theory of evaluative linguistic expressions;
  trichotomous evaluative expressions
- We can distinguish the meaning of simple expressions like *medium, very small, extremely big*, etc., from the meaning of evaluative predications
- Our theory can be further extended to cover fuzzy quantities, negative and compound evaluative expressions
- Our theory can be further extended to intermediate (generalized) quantifiers (*most, a lot of, at least, at most, few*, etc. Generalized Aristotle's syllogisms.