Motivation
Informal principles of the meaning of TEv-expressions
Fuzzy type theory
Formalization

Logical Theory of Evaluative Linguistic Expressions I

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Outline

1. Motivation
2. Informal principles of the meaning of TEv-expressions
3. Fuzzy type theory
4. Formalization
What we are speaking about?

Small, medium, big, twenty five, roughly one hundred, very short, more or less strong, not tall, about twenty five, the sea is deep but not very, roughly small or medium, very roughly strong, weight is small, pressure is very high, extremely rich person
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Importance

Mathematical model of the meaning of evaluative expressions can be ranked among the most important contributions of fuzzy logic

L. A. Zadeh

Why?

- Omnipresent in natural language (people need to evaluate); occur in description of any process, decision situation, procedure, characterization of objects, etc.
- Belong to the agenda of fuzzy logic
  Fuzzy IF-THEN rules, Linguistic variable
- They are present in applications of fuzzy logic
  Fuzzy control, decision making, classification, various industrial applications
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Example of algorithm used by the geologist

1. Ends of sequences are usually rocks types 6, 7, or 8, if they are followed by rock type 1, 2, or 3. If the given rock has lower number followed again by 6, 7, or 8 and it is too thin then it is ignored.

2. Check whether the obtained sequences are sufficiently thick. If the given sequence is too thin then it is joined with the following one, provided that the resulting sequence does not become too thick.

3. If the sequence is too thick then it is further divided: check all rock types 4 and mark them as ends of new sequence provided that the new sequence is not too thin; mark the new sequence only if it is sufficiently thick.
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Grammatical structure

(TEv-expressions)

(i) Simple evaluative expression:

(a) \langle \text{trichotomous evaluative expression} \rangle :=
    \langle \text{linguistic hedge} \rangle \langle \text{TE-adjective} \rangle

\langle \text{TE-adjectives} \rangle: \text{gradable adjectives, adjectives of manner,}
    \text{and possibly some other ones}

\langle \text{linguistic hedge} \rangle — \text{intensifying adverb}

\langle \text{linguistic hedge} \rangle := \emptyset | \langle \text{narrowing adverb} \rangle |
\langle \text{widening adverb} \rangle | \langle \text{specifying adverb} \rangle

(b) \langle \text{fuzzy quantity} \rangle := \langle \text{linguistic hedge} \rangle \langle \text{numeral} \rangle
\langle \text{numeral} \rangle — \text{name of element from the considered scale}
Grammatical structure

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Grammatical structure

(ii) Negative evaluative expression:
not ⟨trichotomous evaluative expression⟩

(iii) Compound evaluative expression:
(a) ⟨trichotomous evaluative expression⟩ or
    ⟨trichotomous evaluative expression⟩
(b) ⟨trichotomous evaluative expression⟩ and/but
    ⟨negative evaluative expression⟩
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TE-adjectives

Antonyms

\[ \langle \text{TE-adjective} \rangle \leftrightarrow \langle \text{antonym} \rangle \]

\begin{align*}
\text{young} & \leftrightarrow \text{old}; \\
\text{ugly} & \leftrightarrow \text{nice}; \\
\text{stupid} & \leftrightarrow \text{clever}; \\
\text{excellent} & \leftrightarrow \text{poor}
\end{align*}

Fundamental evaluative trichotomy

\[ \langle \text{TE-adjective} \rangle \rightleftharpoons \langle \text{middle member} \rangle \rightleftharpoons \langle \text{antonym} \rangle \]

\begin{align*}
\text{young} & \rightleftharpoons \text{medium age} \rightleftharpoons \text{old}; \\
\text{ugly} & \rightleftharpoons \text{normal} \rightleftharpoons \text{nice}; \\
\text{stupid} & \rightleftharpoons \text{medium intelligent} \rightleftharpoons \text{clever}
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Grammatical structure

**Example**

**TE-adjectives:** small, medium, big – canonical  
weak, medium strong, strong; silly, normal, intelligent  
**Fuzzy numbers:** twenty five, roughly one hundred  
**Simple evaluative expressions:** very short, more or less strong, more or less medium, roughly big, about twenty five  
**Negative evaluative expressions:** not short, not very deep  
**Compound evaluative expressions:** roughly small or medium, small but not very (small)
Let $\mathcal{A}$ — evaluative linguistic expression

(a) Evaluative (linguistic) predication

<noun> is $\mathcal{A}$

(b) Abstracted evaluative predication

$X$ is $\mathcal{A}$

$X$ — variable, values are arbitrary elements

(c) Compound evaluative predications:

$\mathcal{A}$ and $\mathcal{B}$  $\mathcal{A}$ or $\mathcal{B}$

(d) Fuzzy IF-THEN rule — abstracted conditional clause

$\mathcal{R} := \text{IF } X \text{ is } \mathcal{A} \text{ THEN } Y \text{ is } \mathcal{B}.$
Let $A$ — evaluative linguistic expression

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Evaluative linguistic predications

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Evaluative linguistic predications

Evaluative predication — general relationship

\[ \mathcal{A} \langle \text{noun} \rangle, \]

small house \( \equiv \) house is small
very tall man \( \equiv \) a man is very tall

Objects named by \( \langle \text{noun} \rangle \) — quite complicated entities

\( \mathcal{A} \) concerns certain feature (or few features) of objects
(attain values from some ordered scale)
Values are evaluated by \( \mathcal{A} \)
Evaluative linguistic predications

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Informal principles of the meaning of TEv-expressions

Any model of the meaning of evaluative linguistic expressions must capture the following concepts:

- **Context** (possible world) — a state of the world at given time moment and place
- **Intension** — a property; it may lead to different truth values in various contexts; *(invariant with respect to various contexts)*
- **Extension** — a class of elements determined by an intension in a given context; *(it does change when changing the context)*
- **Vagueness of the meaning of natural language expressions** is a consequence of the indiscernibility between objects
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(A) *Linguistic context* — nonempty, linearly ordered and bounded scale

Three distinguished limit points: *left bound*, *right bound*, and a *central point*

(B) *Intension* — function from the set of contexts into a set of fuzzy sets.
Each context is assigned a fuzzy set inside it — *extension* of TEv-expression in the given context.
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Informal principles of the meaning of TEv-expressions

(C) Each of the limit points is a starting point of some horizon running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)

Three horizons in each context:
(a) a horizon from the left bound towards central point,
(b) a horizon from the right bound back towards central point,
(c) a horizon from the central point towards both left and right bounds

(D) Each horizon is represented by a fuzzy set determined by a reasoning analogous to that leading to the sorites paradox
Informal principles of the meaning of TEv-expressions

**Motivation**

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(E) *Extension* of each TEv-expression is delineated by a specific horizon obtained by *modification of the horizon*: linguistic hedge

*Shifting* the horizon — moving it closer to, or farther from the limit point (decreasing the truth values)

Small decrease for big truth values; big for small ones

Limit points — *typical values* of TEv-expression
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Informal principles of the meaning of TEv-expressions

(F) Each scale is vaguely partitioned by the **fundamental evaluative trichotomy**

No element of the context falls into extensions of both antonyms
Any element of the scale is contained in the extension of at most two neighboring expressions from the fundamental evaluative trichotomy
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Extensions of evaluative expressions
Fuzzy type theory — syntax

Types: $o$ (truth values), $\epsilon$ (elements)

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\beta \equiv B_\beta$, $\lambda x_\alpha C_\beta$, $\Delta_{oo}$

Interpretation of formulas $A_{\beta\alpha}$: functions $M_\alpha \rightarrow M_\beta$

*Formulas of type $o$ are propositions*  
Interpretation of $A_{o\alpha}$ is a fuzzy subset of $M_\alpha$

Description operator $\iota_\alpha(o_\alpha)$
Fuzzy type theory — syntax

Types: $\mathcal{O}$ (truth values), $\mathcal{E}$ (elements)

Formulas have types: $A_{\alpha} \in \text{Form}_{\alpha}$, $A_{\beta} \equiv B_{\beta}$, $\lambda x_{\alpha} C_{\beta}$, $\Delta_{\mathcal{O}\mathcal{O}}$

Interpretation of formulas $A_{\beta\alpha}$: functions $M_{\alpha} \rightarrow M_{\beta}$

Formulas of type $\mathcal{O}$ are propositions
Interpretation of $A_{\mathcal{O}\alpha}$ is a fuzzy subset of $M_{\alpha}$

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Types: \( o \) (truth values), \( \epsilon \) (elements)

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Interpretation of formulas \( A_\beta^\alpha \): functions \( M_\alpha \rightarrow M_\beta \)

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Fuzzy type theory — semantics I

Frame $\mathcal{M}$

$$\mathcal{I} = \left\langle (M_{\alpha}, =_{\alpha})_{\alpha \in \text{Types}}, \mathcal{L}_{\Delta} \right\rangle$$

(i) $\mathcal{L}_{\Delta}$: IMTL$_{\Delta}$-algebra or Łukasiewicz $\Delta$-algebra

$$\mathcal{L} = \left\langle [0, 1], \lor, \land, \otimes, \Delta, \to, 0, 1 \right\rangle$$

$\lor, \land = \text{minimum, maximum}$

$\otimes = \text{left continuous t-norm}, \quad a \otimes b = 0 \lor (a + b - 1)$

$\to = \text{residuation} \quad a \to b = 1 \land (1 - a + b)$

$\neg a = a \to 0 = 1 - a, \quad \neg \neg a = a$

$\Delta(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$
Fuzzy type theory — semantics I

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$$\mathcal{L} = \langle [0, 1], \lor, \land, \otimes, \Delta, \rightarrow, 0, 1 \rangle$$

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$\rightarrow = \text{residuation} \quad a \rightarrow b = 1 \land (1 - a + b)$

$\neg a = a \rightarrow 0 = 1 - a, \quad \neg \neg a = a$

$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise}. \end{cases}$$
Fuzzy type theory — semantics II

(ii) Fuzzy equality $=_{\alpha}: M_\alpha \times M_\alpha \rightarrow L$

$$[x =_{\alpha} x] = 1 \quad \text{(reflexivity)}$$

$$[x =_{\alpha} y] = [y =_{\alpha} x] \quad \text{(symmetry)}$$

$$[x =_{\alpha} y] \otimes [y =_{\alpha} z] \leq [x =_{\alpha} z] \quad \text{(transitivity)}$$

Example

$$[x =_{\alpha} y] = 0 \lor (1 - |x - y|)$$

$$[x =_{\alpha} y] = \begin{cases} 
1, & \text{if } x = y \\
\frac{1}{v-u} \cdot ((v-x) \land (v-y)) \lor ((x-u) \land (y-u)) & \text{otherwise}
\end{cases}$$
Fuzzy type theory — semantics II

(ii) Fuzzy equality $=_{\alpha} : M_{\alpha} \times M_{\alpha} \rightarrow L$

$[x =_{\alpha} x] = 1$  \hspace{1cm} \text{(reflexivity)}

$[x =_{\alpha} y] = [y =_{\alpha} x]$  \hspace{1cm} \text{(symmetry)}

$[x =_{\alpha} y] \otimes [y =_{\alpha} z] \leq [x =_{\alpha} z]$  \hspace{1cm} \text{(transitivity)}

Example

$[x =_{\alpha} y] = 0 \lor (1 - |x - y|)$

$[x =_{\alpha} y] = \begin{cases} 
1, \text{if } x = y \\
\frac{1}{v-u} \cdot ((v - x) \land (v - y)) \lor ((x - u) \land (y - u)) 
\end{cases}$
Motivation

Informal principles of the meaning of TEv-expressions

Fuzzy type theory

Formalization

Scheme of frame

\[ (M_{\beta\alpha} \subseteq \{ f_{\beta\alpha} \mid f_{\beta\alpha} : M_{\alpha} \rightarrow M_{\beta} \}, =_{\beta\alpha}) \]

\[ (M_{oo} \subseteq \{ g_{oo} \mid g_{oo} : M_0 \rightarrow M_0 \}, =_{oo}) \]

\[ (M_{o\epsilon} \subseteq \{ f_{o\epsilon} \mid f_{o\epsilon} : M_{\epsilon} \rightarrow M_0 \}, =_{o\epsilon}) \]

\[ (M_{\epsilon\epsilon} \subseteq \{ f_{\epsilon\epsilon} \mid f_{\epsilon\epsilon} : M_{\epsilon} \rightarrow M_{\epsilon} \}, =_{\epsilon\epsilon}), \ldots \]

\[ (M_0 = \{ a \mid a \in L \}, \leftrightarrow) \quad (M_{\epsilon} = \{ u \mid \varphi(u) \}, =_{\epsilon}) \]
Interpretation

\[ \mathcal{I}^M(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}} \]

Examples of interpretation:

- \( \mathcal{I}^M(A_\circ) \in L \) – a truth value
- \( \mathcal{I}^M(A_{\circ\epsilon}) \) — fuzzy set in \( M_{\epsilon} \)
- \( \mathcal{I}^M(\iota_{\alpha(\circ\alpha)}) \) is a defuzzification operation
- \( \mathcal{I}^M(\Delta A_\circ) \in \{0, 1\} \) – crisp truth value
Interpretation

\[ I^M(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}} \]

Examples of interpretation:

- \( I^M(A_o) \in L \) – a truth value
- \( I^M(A_{o\epsilon}) \) — fuzzy set in \( M_{\epsilon} \)
- \( I^M(\iota_{\alpha}(o_{\alpha})) \) is a defuzzification operation
- \( I^M(\Delta A_o) \in \{0, 1\} \) – crisp truth value
Interpretation

\[ I^M(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_\beta^{M_\alpha} \]

Examples of interpretation:

- \( I^M(A_o) \in L \) — a truth value
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Interpretation

\[ \mathcal{I}^\mathcal{M}(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^M \]

Examples of interpretation:

- \( \mathcal{I}^\mathcal{M}(A_o) \in L \) — a truth value
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Motivation
Informal principles of the meaning of TEv-expressions
Fuzzy type theory
Formalization

**Formal theory of TEv-expressions**

Construction of formal theory $T^E$ in the language of FTT formalizing general characteristics (A)–(F)

All properties are consequences of 11 special axioms of $T^E$

Formal syntactical proofs of all properties!

**Example (special axioms)**

(EV7) $\Delta((t \Rightarrow u) \& (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u$,

(EV8) $t \equiv t' \& z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z'$,
Formal theory of TEv-expressions

Construction of formal theory \( T^{Ev} \) in the language of FTT formalizing general characteristics (A)–(F)

All properties are consequences of 11 special axioms of \( T^{Ev} \)

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Example (special axioms)

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\begin{align*}
(EV7) & \quad \Delta((t \Rightarrow u) \& (u \Rightarrow z)) \Rightarrow t \sim z \Rightarrow t \sim u, \\
(EV8) & \quad t \equiv t' \& z \equiv z' \Rightarrow t \sim z \Rightarrow t' \sim z',
\end{align*}
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Formal theory of TEv-expressions

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*All properties are consequences of 11 special axioms of $T^{Ev}$*

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**(EV8)** $t \equiv t' \& z \equiv z' \Rightarrow t \sim z \Rightarrow t' \sim z',$
Context

(A) Nonempty, linearly ordered and bounded scale, three distinguished limit points: left bound, right bound, and a central point

Context \( w : [0, 1] \rightarrow M: \)

\[
\begin{align*}
    w(0) &= v_L \\
    w(0.5) &= v_S \\
    w(1) &= v_R
\end{align*}
\]

(\text{left bound})
(\text{central point})
(\text{right bound})

Set of contexts \( W = \{ w \mid w : [0, 1] \rightarrow M \} \)

\( w^{-1} := \exists x \cdot y \equiv wx \)

Linear ordering in each context \( w \)

\[
y \leq_w y' \text{ iff } w^{-1} y \Rightarrow w^{-1} y', \quad y, y' \in w
\]
(A) Nonempty, linearly ordered and bounded scale, three distinguished limit points: *left bound*, *right bound*, and a *central point*

Context \( w : [0, 1] \rightarrow M: \)

\[
\begin{align*}
  w(0) &= \nu_L \\
  w(0.5) &= \nu_S \\
  w(1) &= \nu_R
\end{align*}
\]

(lef bound) (central point) (right bound)

Set of contexts \( W = \{ w \mid w : [0, 1] \rightarrow M \} \)

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Linear ordering in each context \( w \)

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y \leq_w y' \quad \text{iff} \quad w^{-1}y \Rightarrow w^{-1}y', \quad y, y' \in w
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Intension

(B) Function from the set of contexts into a set of fuzzy sets

Scheme of intension
Intension

(B) Function from the set of contexts into a set of fuzzy sets

Scheme of intension

\[ W \rightarrow \mathcal{F}(w ([0,1])) \]

\[ v_L \rightarrow v_S \rightarrow v_R \]

\[ \vdots \]

\[ \vdots \]
(C) Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)
Each of the limit points is a starting point of some *horizon* running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond).
Special fuzzy equality

Fuzzy equality $\sim$ on the set $[0, 1]$ of truth values:

- $[0 \sim 0.5] = 0$
- if $a \leq b \leq c$ then $[a \sim c] \leq [a \sim b]$
- there is $a$ such that $0 < [0 \sim a] < 1$

Example

Standard Łukasiewicz MV-algebra of truth values:

$$[a \sim b] = \frac{0.5 - |a - b|}{0.5}.$$
Horizon

Fuzzy equality induced by $\sim$

\[ y \approx_w y' := \text{iff } w^{-1}y \sim w^{-1}y', \quad x, y \in w \]

Three horizons

\[ LH(a) = [0 \sim a], \quad LH(w x) = [v_L \approx_w x] \]
\[ MH(a) = [0.5 \sim a], \quad MH(w x) = [v_S \approx_w x] \]
\[ RH(a) = [1 \sim a], \quad RH(w x) = [v_R \approx_w x] \]
Properties of horizon

(D) Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

Special context $w_N$: $v_L = 0$, $v_S = p$, $v_R = q$

$FN(n) = [0 \approx_{w_N} n]$ — finite numbers do not form a heap

Theorem

(a) $FN(0) = 1$
(b) $p \leq n \Rightarrow FN(n) = 0$
(c) $FN(n + 1) \leq FN(n)$
(d) No $n$ such that $FN(n) = 1$ and $FN(n + 1) = 0$
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Hedge: shift of the horizon

\((E)\) Extension of each TEv-expression is delineated by modification (shifting) of the horizon

Modification: linguistic hedge

Hedges — horizon modifications

\[ \nu : [0, 1] \rightarrow [0, 1] \]
Hedge: shift of the horizon

(E) Extension of each TEv-expression is delineated by modification (shifting) of the horizon
Modification: linguistic hedge

Hedges — horizon modifications

\[ \nu : [0, 1] \longrightarrow [0, 1] \]
Formalization of intension and extension

Intension — a function $W \rightarrow \mathcal{F}(w([0, 1]))$

(i) **S-intension:**

$$Sm\nu = \lambda w \lambda x \cdot \nu(LH(w^{-1}x))$$

(ii) **M-intension:**

$$Me\nu = \lambda w \lambda x \cdot \nu(MH(w^{-1}x))$$

(iii) **B-intension:**

$$Bi\nu = \lambda w \lambda x \cdot \nu(RH(w^{-1}x))$$

Extension of an evaluative predication in the context $w$

$$(Ev\nu)w \subseteq w([0, 1])$$
Formalization of intension and extension

**Intension** — a function \( \mathcal{W} \rightarrow \mathcal{F}(w([0, 1])) \)

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**Extension** of an evaluative predication in the context \( w \)

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Motivation Informal principles of the meaning of TEv-expressions Fuzzy type theory Formalization

Fundamental evaluative trichotomy

(F) Each scale is vaguely partitioned by the fundamental evaluative trichotomy

Special hedge $\bar{\nu}$

$$\overline{Sm}(w) = \lambda x \cdot \bar{\nu}(LH(w^{-1}x))$$
$$\overline{Me}(w) = \lambda x \cdot \bar{\nu}(MH(w^{-1}x))$$
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Fundamental evaluative trichotomy

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Special hedge $\tilde{\nu}$

$$\overline{Sm}(w) = \lambda x \cdot \tilde{\nu}(LH(w^{-1}x))$$
$$\overline{Me}(w) = \lambda x \cdot \tilde{\nu}(MH(w^{-1}x))$$
$$\overline{Bi}(w) = \lambda x \cdot \tilde{\nu}(RH(w^{-1}x))$$
Fundamental evaluative trichotomy
Intensions of evaluative predications

\langle\text{linguistic hedge}\rangle \mapsto \nu \\
- \text{Int}(X \text{ is } \langle\text{linguistic hedge}\rangle \text{ small}) := Sm\nu \\
- \text{Int}(X \text{ is } \langle\text{linguistic hedge}\rangle \text{ medium}) := Me\nu \\
- \text{Int}(X \text{ is } \langle\text{linguistic hedge}\rangle \text{ big}) := Bi\nu \\

Special case:

\begin{align*}
\text{Int}(X \text{ is small}) & := \overline{Sm} \\
\text{Int}(X \text{ is medium}) & := \overline{Me} \\
\text{Int}(X \text{ is big}) & := \overline{Bi}
\end{align*}

Theorem

Formal theory $T^\text{Ev}$ of evaluative linguistic expressions is consistent
Intensions of evaluative predications

\[ \langle \text{linguistic hedge} \rangle \mapsto \nu \]

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\[
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\end{align*}
\]

**Theorem**

*Formal theory T^{Ev} of evaluative linguistic expressions is consistent*
Conclusions

- Development of a comprehensive logical theory of evaluative linguistic expressions; trichotomous evaluative expressions
- We can distinguish the meaning of simple expressions like medium, very small, extremely big, etc., from the meaning of evaluative predications
- Our theory can be further extended to cover fuzzy quantities, negative and compound evaluative expressions
- Our theory can be further extended to intermediate (generalized) quantifiers (most, a lot of, at least, at most, few, etc. Generalized Aristotle’s syllogisms.
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