Logical Theory of Evaluative Linguistic Expressions II

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Praha, 29. 10. 2007
Outline

1. Fuzzy type theory

2. Formalization of TEv-expression
What we are speaking about?

- **Evaluative linguistic expressions:**
  small, medium, big, twenty five, roughly one hundred, very short, more or less strong, not tall, about twenty five, the sea is deep but not very, roughly small or medium, very roughly strong

- **Evaluative linguistic predications:**
  weight is small, pressure is very high, extremely rich person
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Fuzzy type theory — syntax

- B. Russel

**Types**

**Elementary types:** $\scriptstyle o$ (truth values), $\epsilon$ (objects)

**Composed types:** $\beta\alpha$

Each formula has a certain type: $A_\alpha \in Form_\alpha$

Fuzzy equality: $A_\beta \equiv B_\beta$

$\lambda x_\alpha C_\beta$ — formula of type $\beta\alpha$ (*lambda term*)

Delta connective: $\Delta_{oo}$ (*surely*)

Description operator $\iota_{\alpha(\omega\alpha)}$

**Formulas of type $o$ are propositions**
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### Types

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Each formula has a certain type: $A_α \in Form_α$

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Types

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- Delta connective: \( \Delta_{\alpha\alpha} \) (surely)

Description operator \( \iota_\alpha(\alpha\alpha) \)

Formulas of type \( \alpha \) are propositions
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**Types**

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Formulas of type \( o \) are propositions
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Types

Elementary types: $\circ$ (truth values), $\epsilon$ (objects)
Composed types: $\beta_\alpha$

Each formula has a certain type: $A_\alpha \in Form_\alpha$
Fuzzy equality: $A_\beta \equiv B_\beta$
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Delta connective: $\Delta_{oo}$ *(surely)*
Description operator $\iota_\alpha(\circ_\alpha)$

**Formulas of type $\circ$ are propositions**
Logical connectives

- **Equivalence:** $A_o \equiv B_o$  \textbf{basic connective!}
- **Implication:** $A_o \Rightarrow B_o$
- **Disjunction:** $A_o \lor B_o$
- **Conjunction:** $A_o \land B_o$, interpreted by minimum ($\land$) phrasal conjunction
- **Strong conjunction:** $A_o \& B_o$, interpreted by $\otimes$ sentential conjunction
- **Delta connective (surely):** $\Delta A_o$

*Local character of conjunction:* resulting truth degree depends on the meaning of conjuncts

*(slow and safe car; big and beautiful house)*

Modus ponens requires strong conjunction

$$
\frac{a/A_o, c/A_o \Rightarrow B_o}{a \otimes c/B_o}
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$$
\begin{align*}
  & a/A_o, c/A_o \Rightarrow B_o \\
  \hline
  & a \otimes c/B_o
\end{align*}
$$
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Structure of truth values

Łukasiewicz $\Delta$-algebra ($\mathcal{L}_\Delta$) (IMTL$\Delta$-algebra)

$$\mathcal{L} = \langle [0, 1], \lor, \land, \otimes, \Delta, \rightarrow, 0, 1 \rangle$$

$\lor, \land$ minimum, maximum

$$a \otimes b = 0 \lor (a + b - 1), \quad (\otimes = \text{left continuous t-norm})$$

$$a \rightarrow b = 1 \land (1 - a + b) \quad (\rightarrow = \text{residuation})$$

$$\neg a = a \rightarrow 0 = 1 - a, \quad \neg \neg a = a$$

$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$a \leftrightarrow b = (a \rightarrow b) \land (b \rightarrow a)$$
Fuzzy equality

Fuzzy relation $=^{\alpha}$: $M_{\alpha} \times M_{\alpha} \rightarrow L$

\[
[x =^{\alpha} x] = 1 \quad \text{(reflexivity)}
\]
\[
[x =^{\alpha} y] = [y =^{\alpha} x] \quad \text{(symmetry)}
\]
\[
[x =^{\alpha} y] \otimes [y =^{\alpha} z] \leq [x =^{\alpha} z] \quad \text{(transitivity)}
\]

Example

\[
[x =^{\alpha} y] = 0 \lor (1 - |x - y|)
\]
\[
[x =^{\alpha} y] = \begin{cases} 
1, \text{if } x = y \\
\frac{1}{v-u} \cdot ((v-x) \land (v-y)) \lor ((x-u) \land (y-u)) 
\end{cases}
\]
Fuzzy equality

Fuzzy relation $=_{\alpha}: M_{\alpha} \times M_{\alpha} \rightarrow L$

$[x =_{\alpha} x] = 1$ \hspace{1cm} (reflexivity)

$[x =_{\alpha} y] = [y =_{\alpha} x]$ \hspace{1cm} (symmetry)

$[x =_{\alpha} y] \otimes [y =_{\alpha} z] \leq [x =_{\alpha} z]$ \hspace{1cm} (transitivity)

Example

$[x =_{\alpha} y] = 0 \lor (1 - |x - y|)$

$[x =_{\alpha} y] = \begin{cases} 1, \text{ if } x = y \\ \frac{1}{v-u} \cdot ((v-x) \land (v-y)) \lor ((x-u) \land (y-u)) \end{cases}$
Semantics – frame

Frame $\mathcal{M}$

$\mathcal{I} = \langle (M_\alpha, =\alpha)_{\alpha \in \text{Types}}, \mathcal{L}_\Delta \rangle$

$(M_o = \{a \mid a \in L\}, \leftrightarrow)$ \quad $(M_\epsilon = \{u \mid \varphi(u)\}, =\epsilon)$

$(M_{oo} \subseteq \{g_{oo} \mid g_{oo} : M_o \rightarrow M_o\}, =_{oo})$

$(M_{o\epsilon} \subseteq \{f_{o\epsilon} \mid f_{o\epsilon} : M_o \rightarrow M_\epsilon\}, =_{o\epsilon})$

$(M_{\epsilon\epsilon} \subseteq \{f_{\epsilon\epsilon} \mid f_{\epsilon\epsilon} : M_\epsilon \rightarrow M_\epsilon\}, =_{\epsilon\epsilon}), \ldots$

$\vdots$

$(M_{\beta\alpha} \subseteq \{f_{\beta\alpha} \mid f_{\beta\alpha} : M_\alpha \rightarrow M_\beta\}, =_{\beta\alpha})$

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Fuzzy type theory

Formalization of TEv-expression

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Interpretation

\[ \mathcal{I}^M(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M^{M_{\alpha}}_{\beta} \]

Interpretation of formulas \( A_{\beta\alpha} \): functions \( M_{\alpha} \rightarrow M_{\beta} \)

**Example**

- \( \mathcal{I}^M(A_o) \in L \) – a truth value
- \( \mathcal{I}^M(A_{o\epsilon}) \) — fuzzy set in \( M_{\epsilon} \)
- \( \mathcal{I}^M(A_{(o\beta)\alpha}) \) — fuzzy relation in \( M_{\alpha} \times M_{\beta} \)
- \( \mathcal{I}^M(\iota_{\alpha(o\alpha)}) \) — defuzzification operation
- \( \mathcal{I}^M(\Delta A_o) \in \{0, 1\} \) – crisp truth value
Interpretation

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**Interpretation**

$$\mathcal{I}^M(A_{\beta\alpha}) = f_{\beta\alpha} \in M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}}$$

Interpretation of formulas $A_{\beta\alpha}$: functions $M_{\alpha} \rightarrow M_{\beta}$

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**Interpretation**

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Logical axioms

17 axioms

- **Equality axioms**
  
  (FT1) $\Delta(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta \alpha} x_\alpha \equiv f_{\beta \alpha} y_\alpha)$

- **Truth values axioms**
  
  (FT6) $(x_o \equiv y_o) \equiv ((x_o \Rightarrow y_o) \land (y_o \Rightarrow x_o))$

- **Delta axioms**
  
  (FT5) $(g_{oo}(\Delta x_o) \land g_{oo}(\neg \Delta x_o)) \equiv (\forall y_o)g_{oo}(\Delta y_o)$

- **Predicate axiom**
  
  (FT16) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$

  $x_\alpha$ is not free in $A_o$

- **Axiom of descriptions**
  
  (FT17) $\iota_{\epsilon(o\epsilon)}(E_{(o\epsilon)\epsilon} y_\epsilon) \equiv y_\epsilon$
Inference rules and provability

Rule (R)

Let $A_\alpha \equiv A'_\alpha$ and $B \in \text{Form}_o$. Then we infer $B'$ where $B'$ comes from $B$ by replacing one occurrence of $A_\alpha$, which is not preceded by $\lambda$, by $A'_\alpha$.

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Let $A_o \in \text{Form}_o$ be a formula. Then from $A_o$ infer $\Delta A_o$.

Formal theory $T$ of FTT is a set of formulas of type $o$.

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Completeness

**Theorem**

(a) A theory $T$ of fuzzy type theory is consistent iff it has a general model $M$.

(b) For every theory $T$ of the fuzzy type theory and a formula $A_o$

$$T \vdash A_o \iff T \models A_o.$$ 

**Claim**

All essential properties of vague predicates are formally expressible in FTT and so, they have a many-valued model.
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We construct a formal theory $T^E_v$ in the language of FTT formalizing general 6 characteristics

*All properties are consequences of 11 special axioms of $T^E_v$*

*Formal syntactical proofs of all properties!*
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Axioms of $T^{Ev}$

(EV1) $(\exists z)\Delta(\neg z \equiv z)$

(EV2) $(\bot \equiv w^{-1}\bot_w) \land (\dagger \equiv w^{-1}\dagger_w) \land (\top \equiv w^{-1}\top_w)$

(EV3) $t \sim t$

(EV4) $t \sim u \equiv u \sim t$

(EV5) $t \sim u \land u \sim z \Rightarrow t \sim z$

(EV6) $\neg(\bot \sim \dagger)$

(EV7) $\Delta((t \Rightarrow u) \land (u \Rightarrow z)) \Rightarrow \cdot t \sim z \Rightarrow t \sim u$

(EV8) $t \equiv t' \land z \equiv z' \Rightarrow \cdot t \sim z \Rightarrow t' \sim z'$

(EV9) $(\exists u)\hat{\Upsilon}(\bot \sim u) \land (\exists u)\hat{\Upsilon}(\dagger \sim u) \land (\exists u)\hat{\Upsilon}(\top \sim u)$

(EV10) $NatHedge \nu \land (\exists \nu')(\exists \nu')(Hedge \nu \land Hedge \nu' \land (\nu_1 \leq \nu \land \nu \leq \nu_2))$

(EV11) $(\forall z)((\gamma \nu(LH z)) \lor (\gamma \nu(MH z)) \lor (\gamma \nu(RH z)))$
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Context

**(A)** Nonempty, linearly ordered and bounded scale, three distinguished limit points: *left bound, right bound, and a central point*

Context \( w_{\alpha_0} \) \( \mathcal{I}(w_{\alpha_0}) = w : [0, 1] \rightarrow M: \)

\[
\begin{align*}
  w(0) &= v_L \
  w(0.5) &= v_S \
  w(1) &= v_R
\end{align*}
\]

(\text{left bound}) \hspace{2cm} (\text{central point}) \hspace{2cm} (\text{right bound})

Set of contexts \( W = \{ w \mid w : [0, 1] \rightarrow M \} \)

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Intension

(\textbf{B}) Function from the set of contexts into a set of fuzzy sets

\[ \text{Int}(\mathcal{A}) = \lambda w \lambda x (Aw)x \quad \mathcal{I}(\text{Int}(\mathcal{A})) : \mathcal{W} \rightarrow \mathcal{F}(w([0, 1])) \]
Intension

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Scheme of intension
(C) Each of the limit points is a starting point of some horizon running from it in the sense of the ordering of the scale towards the next limit point (the horizon vanishes beyond)
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We introduce a special fuzzy equality $\sim$ on the set $[0, 1]$ of truth values

**Example**

Standard Łukasiewicz MV-algebra of truth values:

$$[a \sim b] = \frac{0.5 - |a - b|}{0.5}.$$ 

Fuzzy equality induced by $\sim$

$$y \approx_w y' := \text{iff } w^{-1}y \sim w^{-1}y', \quad x, y \in w$$
Three horizons

\[ LH(a) = [0 \sim a], \quad LH(w \ x) = [v_L \approx_w x] \]
\[ MH(a) = [0.5 \sim a], \quad MH(w \ x) = [v_S \approx_w x] \]
\[ RH(a) = [1 \sim a], \quad RH(w \ x) = [v_R \approx_w x] \]
(D) Each horizon is represented by a special fuzzy set determined by a reasoning analogous to that leading to the sorites paradox.

**Sorites paradox**
One grain does not make a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.

**Falakros paradox**
A man with no hair is bald. A man with one hair more than a bald man is still bald. Consequently, all men are bald.
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Sorites in fuzzy logic

New predicate $\mathbb{F}_N(n)$: “$n$ is small”; “$n$ is feasible”; “$n$ is finite”

Axioms

- “there is a small number” — valid
- “if $n$ is small then $n + 1$ is also small” — practically valid!
- “there is a number not being small” — valid

No contradiction!
(possible in classical logic)
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Sorites in the theory of TEv-expression

Special context $w_N$:

$$\nu_L = 0, \quad \nu_S = p, \quad \nu_R = q$$

$\approx_{w_N}$ – induced fuzzy equality in the context $w_N$

$$FN := \lambda n \cdot 0 \approx_{w_N} n$$

Finite numbers do not form a heap

Interpretation: fuzzy set

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Theorem

(a) \( \vdash \Delta FN 0, \)

(b) \( \vdash n \in w_N \& \Delta(p \leq n) \Rightarrow \neg FN n \)

(c) \( \vdash m \in w_N \& n \in w_N \& \Delta(m \leq n) \Rightarrow (FNn \Rightarrow FNm) \)

(d) \( \vdash \neg(\exists n)(n \in w_N \& \Delta FN n \& \Delta \neg FN(n + 1)) \)

(e) \( \vdash n \in w_N \Rightarrow (FNn \Rightarrow (n \approx w_N n + 1) \Rightarrow FN(n + 1)) \)
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(a) ⊢ ΔFN 0,

(b) ⊢ n ∈ w_N & Δ(p ≤ n) ⇒ ¬FN n

(c) ⊢ m ∈ w_N & n ∈ w_N & Δ(m ≤ n) ⇒ (FN n ⇒ FN m)

(d) ⊢ ¬(∃n)(n ∈ w_N & ΔFN n & Δ¬FN(n + 1))

(e) ⊢ n ∈ w_N ⇒ (FN n ⇒ (n ≈_{w_N} n + 1) ⇒ FN(n + 1))
Hedge: shift of the horizon

(E) Extension of each TEv-expression is delineated by modification (shifting) of the horizon
Modification: linguistic hedge

Hedges — horizon modifications

ν : [0, 1] → [0, 1]
Hedge: shift of the horizon

(E) Extension of each TEv-expression is delineated by modification (shifting) of the horizon
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Hedges — horizon modifications

\[ \nu : [0, 1] \rightarrow [0, 1] \]
Each scale is vaguely partitioned by the fundamental evaluative trichotomy
Intensions of evaluative predications

\langle \text{linguistic hedge} \rangle \mapsto \nu

- \text{Int}(X \text{ is } \langle \text{linguistic hedge} \rangle \text{ small}) :=
  \[ Sm\nu = \lambda w \lambda x \cdot \nu(LH(w^{-1}x)) \]

- \text{Int}(X \text{ is } \langle \text{linguistic hedge} \rangle \text{ medium}) :=
  \[ Me\nu = \lambda w \lambda x \cdot \nu(MH(w^{-1}x)) \]

- \text{Int}(X \text{ is } \langle \text{linguistic hedge} \rangle \text{ big}) :=
  \[ Bi\nu = \lambda w \lambda x \cdot \nu(RH(w^{-1}x)) \]

\text{Extension} of an evaluative predication in the context \( w \)

\((Ev\nu)w \subseteq w([0, 1])\)
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  \text{Int}(X \text{ is } \langle \text{linguistic hedge} \rangle \text{ big}) :=
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  Bi_{\nu} &= \lambda w \lambda x \cdot \nu(RH(w^{-1}x))
  
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Extension of an evaluative predication in the context \( w \)

\( (Ev\nu)_w \subseteq w([0, 1]) \)
**Fundamental evaluative trichotomy**

*Intensions of “small” and “big” have in each context the basic property of antonyms*

**Theorem**

For all \( w \) and \( x \in w \)

\[ \vdash \Delta \neg((Smw) x \land (Biw)x) \]

*Each context is fully covered by the fundamental evaluative trichotomy*

**Theorem**

For all \( w \) and \( x \in w \)

\[ \vdash (Evalw x Sm) \lor (Evalw x Me) \lor (Evalw x Sm) \]
Fundamental evaluative trichotomy

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Theorem

For all $w$ and $x \in w$

$$\vdash \Delta \neg((\overline{Sm} w) x \land (\overline{Bi} w) x)$$

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\[ \vdash (Eval w x \overline{Sm}) \lor (Eval w x \overline{Me}) \lor (Eval w x \overline{Sm}) \]
Vagueness of “small” and “big”

**Theorem**

*No last small natural number*

\[ \vdash \neg (\exists n)(n \in w_N \land \Delta(Sm \nu) w_N n \land \Delta \neg (Sm \nu) w_N (n + 1)) \]

**Theorem**

- **There is no last small value**
  \[ \vdash \neg (\exists x)(\forall y)(\Delta(Sm \nu) w x \land (x <_w y \Rightarrow \Delta \neg (Sm \nu) w y)) \]

- **There is no first big value**
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*Formal theory \( T^{Ev} \) of evaluative linguistic expressions is consistent*
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Theorem

No last small natural number
⊢ ¬(∃n)(n ∈ wN & ∆(Sm ν)wNn & ∆¬(Sm ν)wN(n + 1))

Theorem

There is no last small value
⊢ ¬(∃x)(∀y)(∆(Sm ν)wx & (x <w y ⇒ ∆¬(Sm ν)wy))

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